

## DISCUSSION

### DISCUSSION OF "ON FINITE PLASTIC FLOWS OF COMPRESSIBLE MATERIALS WITH INTERNAL FRICTION"

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The purpose of this discussion is to clarify the relationships between constitutive laws proposed by Nemat-Nasser and Shokooch[1] and by Rudnicki and Rice[2] to model the behavior of pressure-sensitive materials exhibiting inelastic volume changes.

Although Nemat-Nasser and Shokooch[1] state that their constitutive law generalizes that proposed by Rudnicki and Rice[2], this claim is based on a misinterpretation of the Rudnicki and Rice law. In fact, the law proposed in [1] is a specific example of the smooth yield surface, isotropic hardening laws suggested in [2]. (This fact does not, of course, diminish the contribution of Nemat-Nasser and Shokooch[1] in exploring in detail the application of this form of constitutive law to the behavior of granular materials.) Furthermore, in comparing their constitutive law with a law proposed by Rudnicki and Rice[2] to model approximately the response at a yield surface vertex, Nemat-Nasser and Shokooch[1] incorrectly describe the latter law as "deduced by generalizing linear elasticity equations." In fact, the law was introduced to overcome the deficiency of formulations incorporating a smooth plastic potential surface, like that suggested in [1], in describing the stiffness of response to abrupt changes in the pattern of deformation. As noted by Storen and Rice[4], the modified Rudnicki and Rice law can be interpreted as a deformation plasticity theory. A variety of recent studies involving the development of localized plastic deformation, e.g. [3, 4, 6, 7] have employed deformation theories to approximate the response at a yield surface vertex. Because of the recent widespread use of these theories, a clarified treatment is warranted even though the application of the constitutive law in [1] (to model axisymmetric compression data on sand and crushed Westerly granite) is one for which yield surface vertex effects are likely to be insignificant and an isotropic hardening law will be adequate.

Rudnicki and Rice[2], motivated by the behavior of fissured rock masses in compression, suggest several constitutive formulations for pressure-sensitive materials in which inelastic deformation involves volume change. One, a simple generalization of the Prandtl-Reuss equations often used in metal plasticity, has the following form (eqn 10) of [2]):

$$2D'_{ij} = \frac{\dot{\sigma}'_{ij}}{G} + \frac{1}{H} \frac{\sigma'_{ij}}{\tau} \left[ \frac{\sigma'_{kk} \dot{\sigma}'_{kk}}{2\tau} + \frac{\mu}{3} \dot{\sigma}'_{kk} \right]$$

$$D_{kk} = \frac{\dot{\sigma}'_{kk}}{3K} + \frac{\beta}{H} \left[ \frac{\sigma'_{kk} \dot{\sigma}'_{kk}}{2\tau} + \frac{\mu}{3} \dot{\sigma}'_{kk} \right] \quad (1)$$

where  $D_{ij}$ , the rate of deformation tensor, and  $\dot{\sigma}'_{ij}$ , the Jaumann (co-rotational) rate of Cauchy (true) stress, are defined as in [1], the superposed prime denotes the deviatoric part and  $\tau = (1/2 \sigma'_{ij} \sigma'_{ij})^{1/2}$  is the Mises equivalent shear stress.  $G$  and  $K$  are incremental elastic shear and bulk moduli, respectively;  $H$  is a plastic modulus termed hardening if positive or softening if negative;  $\mu$  is a friction coefficient expressing the dependence of the yield (flow) stress in shear,  $\tau_{flow}$ , on the hydrostatic stress, that is,

$$\mu = \frac{d\tau_{flow}}{d\sigma}$$

and  $\beta$  is a dilatancy (compaction) factor equal to the ratio of inelastic volume strain increment

to inelastic shear strain increment, that is,

$$\beta = D_{kk}^P / [2D_{ij}^P D_{ij}^P]^{1/2}$$

where the superposed  $P$  denotes the inelastic portion. The modulus  $H$  is related to the slope of the shear stress  $\tau$  versus shear strain  $\gamma$  curve at constant hydrostatic stress by

$$\frac{d\tau}{d\gamma} = \frac{H}{1 + H/G} \quad \text{for } \sigma = \text{constant.}$$

Equations (1) are appropriate for deformation increments tending to cause continued inelastic deformation, that is, increments tending to make  $d\tau > \mu d(\sigma_{kk}/3)$ . The second term in each equation is to be dropped for elastic unloading, that is, for increments tending to make  $d\tau < \mu d(\sigma_{kk}/3)$ .

Nemat-Nasser and Shokooh (p. 495, paragraph 2, in [1]) incorrectly state that  $\mu$  and  $\beta$  are constants in the formulation of [2]. However,  $\mu$  and  $\beta$ , and indeed all the constitutive parameters, are restricted only in that they depend on the current state and not on deformation or stress increments (Section 3.1 of [2]). Consequently, the constitutive parameters may depend on any quantity characterizing the current state. Nemat-Nasser and Shokooh[1] make the particular choice of hydrostatic stress, plastic volumetric strain  $\Delta$ , and total distortional work  $\xi$ . Specifically, the yield function and plastic potential are given in (2.7) and (2.5) of [1] by

$$f \equiv \tau - F(\sigma_{kk}, \Delta, \xi)$$

and

$$g \equiv \tau + G(\sigma_{kk}, \Delta, \xi),$$

respectively. As shown in [1], the friction coefficient and dilatancy factor can be identified as

$$\mu = - \frac{\partial F}{\partial(\sigma_{kk}/3)}$$

and

$$\beta = \frac{\partial G}{\partial(\sigma_{kk}/3)}.$$

Consequently, the constitutive law suggested in [1] falls within the class of the isotropic hardening laws proposed by Rudnicki and Rice[2]. Rudnicki and Rice[2] do not, however, recommend any explicit dependence of  $\mu$  and  $\beta$  on deformation. Nemat-Nasser and Shokooh[1] do deduce a structure for this dependence in the case of granular materials and they show that this has interesting and useful consequences for describing the behavior of such materials. Thus, the constitutive law discussed in [1] provides a specific example of the application of the class of laws proposed in [2] to the behavior of granular materials.

The constitutive law (1), and other formulations incorporating a smooth plastic potential, predicts purely elastic response for the portion of the stress increment directed tangential to the smooth yield surface. For example, consider the state of axisymmetric deformation discussed in Section 4 of [1]:

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_{33} = \sigma_2, \quad \text{all other } \sigma_{ij} = 0, \quad \sigma_1 > \sigma_2$$

with stresses taken as positive in compression. The response to an increment of shear is purely elastic:

$$2D_{ij} = \frac{\tau}{G} \quad (i \neq j).$$

In contrast, detailed microstructural models of material behavior[2,9,10] predict the existence of a sharp vertex on the current yield surface. Consequently, these models predict purely

elastic response only for stress increments directed within the cone of the vertex and, in particular, a stress increment directed tangential to what would be the smooth yield surface through the same point causes plastic deformation. Although experimental observations (e.g. [8]) are inconclusive about the existence of yield surface vertices, they do corroborate the predictions of microstructural models that isotropic hardening formulations overestimate the stiffness of response to abrupt changes in the pattern of deformation or strongly non-proportional loading histories.

In order to remedy this deficiency of [1], at least in an approximate way, Rudnicki and Rice[2] introduced a second plastic modulus  $H_1$  to govern the portion of the stress increment directed tangential to what would be the smooth yield surface through the same stress point. With this modification, the deviatoric portion of the rate of deformation becomes (see (13) of [2])

$$2D'_{ij} = \frac{\overset{*}{\sigma}'_{ij}}{G} + \frac{1}{H} \frac{\sigma'_{ij}}{\tau} \left[ \frac{\sigma'_{kl} \overset{*}{\sigma}_{kl}}{2\tau} + \mu \frac{\overset{*}{\sigma}_{kk}}{3} \right] + \frac{1}{H_1} \left[ \overset{*}{\sigma}'_{ij} - \frac{\sigma'_{ij} \sigma'_{kl}}{2\tau^2} \overset{*}{\sigma}_{kl} \right]. \quad (3)$$

For stress increments directed normal to the projection of the yield surface in the deviatoric plane, that is, for

$$\overset{*}{\sigma}'_{ij} \propto \sigma'_{ij} \quad (4)$$

there is no contribution from the second term in (3) and (3) reduces to (1). In contrast to (1), however, (3) predicts that the response to an increment of shear applied to a state of axisymmetric deformation is

$$D_{ij} = \overset{*}{\sigma}_{ij} \left( \frac{1}{G} + \frac{1}{H_1} \right) \quad (i \neq j). \quad (5)$$

Equation (3) was intended to model the response at a yield surface vertex for small deviations from "straightahead" loading given by (4). (For small deviations, the trace of  $D_{ij}$  is still given by the second of (1).) Storen and Rice[4] have interpreted (3) in terms of a finite strain version of  $J_2$  deformation theory of plasticity. (See [7] for another version.) Christoffersen and Hutchinson[3] have formulated a more elaborate corner theory that allows continuous variation of the stiffness of response with deviations from straightahead loading.

Nemat-Nasser and Shokooh[1] also introduce a second plastic modulus: they write

$$H = h + h_1 \quad (6)$$

and incorporate into  $h_1$  all changes of  $H$  due to changes in density or hydrostatic stress. This modification is, however, completely unrelated to the introduction of  $H_1$  to approximate the behavior at a yield surface vertex and, consequently, the comparison in [1] of this version of (1) with (3) is misleading. Nemat-Nasser and Shokooh[1] state that "A basic difference between the equations is that in (3.6) (eqn (3) here) the introduction of the second modulus also affects the elastic response, whereas in (3.5) (the version of (1) incorporating (6) in [1]) the elastic and plastic parts are completely kept apart." Of course, elastic and plastic portions of the response are also "kept apart" in (3) and the quoted statement obscures the motivation for the introduction of the second modulus in [2].

Furthermore, Nemat-Nasser and Shokooh[1] state that "eqn. (3.5) presents a proper flow theory plasticity whereas (3.6) is deduced by generalizing linear elasticity equations" and thereby imply that (3.6) (eqn (3) here) is not a proper plasticity theory. As already mentioned, (3) can be interpreted as a finite strain version of  $J_2$  deformation theory. Although deformation theories are not suitable when unloading occurs, they have proved to be useful in approximating the response at a yield surface vertex to abrupt changes in the pattern of deformation[3, 7, 11]. Moreover, deformation theories are well-known to give much better agreement with experiments than flow theories for problems involving abrupt deviations from

proportional loading, for example, buckling problems[5]. Consequently, for such problems deformation theories offer a more suitable idealization of actual behavior than flow theories. If the plastic potential in the version of (1) advanced by Nemat-Nasser and Shokooh[1] is assumed to be a smooth function, this law does not include the possibility of approximating yield surface vertex effects.

In addition, Nemat-Nasser and Shokooh[1] state that in (3) "the effective elastic shear modulus is predicted to increase with plastic deformation, and hence the relative values of the strain rate components depend on relative values of the stress rates." The latter statement is, however, the usual interpretation of what is meant by "vertex effects." Moreover, even in the smooth yield surface version of the constitutive law (1), the elastic moduli are incremental moduli and hence may change with deformation. Rudnicki and Rice[2] remark that the elastic moduli of brittle rock will, in contrast to the behavior of metals, decrease by as much as 50% with increasing inelastic deformation due to microcracking and sliding on fissure surfaces. In any case, the effective shear modulus in (3) is given by the coefficient in (5). Because  $H_1$  has an approximate interpretation as a plastic secant modulus, the effective shear modulus actually decreases with plastic deformation.

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